FUNCTIONS OF REPRESENTATIONS OF THE CLASS 1 ON THE HOMOGENEOUS SPACES OF THE DE SITTER GROUP

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A starting point of this research is an analogue between universal coverings of the Lorentz and de Sitter groups, which was first established by Takahashi [1] (see also the work of Ström [2]). Namely, the universal covering of $SO_0(1,4)$ is $\mathbf{Spin}_+(1,4) \simeq Sp(1,1)$ and the spinor group $\mathbf{Spin}_+(1,4)$ is described in terms of 2×2 quaternionic matrices. Spherical functions on the group $SO_0(1,4)$ are understood as functions of representations of the class 1 realized on the homogeneous spaces of $SO_0(1,4)$. A list of homogeneous spaces of $SO_0(1,4)$, including symmetric Riemannian and non-Riemannian spaces, consists of the group manifold $\mathfrak{S}_{10}$ of $SO_0(1,4)$, two-dimensional quaternion sphere S^q_2, four-dimensional hyperboloid $H^4 \sim SO_0(1,4)/SO(4)$, three-dimensional real sphere $S^3 \sim SO(4)/SO(3)$ and a two-dimensional real sphere $S^2 \sim SO(3)/SO(2)$.

Using the universal covering $\mathbf{Spin}_+(1,4) \simeq Sp(1,1)$ of $SO_0(1,4)$, we can write a first Casimir operator F on the group manifold $\mathfrak{S}_{10}$,

$$-F = \frac{\partial^2}{\partial \theta^{q2}} + \cot\theta^q \frac{\partial}{\partial\theta^q} + \frac{1}{\sin^2\theta^q}\frac{\partial^2}{\partial\varphi^{q2}} - \frac{2\cos\theta^q}{\sin^2\theta^q}\frac{\partial^2}{\partial\varphi^q\partial\psi^q_1} + \cot^2\theta^q\frac{\partial^2}{\partial\psi_1^{q2}} + \frac{\partial^2}{\partial\psi^{q2}}, \tag{1}$$

where

$$\begin{array}{rclcrcl}
\frac{\partial}{\partial\theta^q} &=& \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi} + \mathbf{i}\frac{\partial}{\partial\tau}, & & \frac{\partial}{\partial\dot{\theta}^q} &=& \frac{\partial}{\partial\theta} - \frac{\partial}{\partial\phi} - \mathbf{i}\frac{\partial}{\partial\tau},\\
\frac{\partial}{\partial\varphi^q} &=& \frac{\partial}{\partial\varphi} + \mathbf{i}\frac{\partial}{\partial\epsilon} + \mathbf{j}\frac{\partial}{\partial\varsigma}, & & \frac{\partial}{\partial\dot{\varphi}^q} &=& \frac{\partial}{\partial\varphi} - \mathbf{i}\frac{\partial}{\partial\epsilon} - \mathbf{j}\frac{\partial}{\partial\varsigma},\\
\frac{\partial}{\partial\psi^q} &=& \frac{\partial}{\partial\psi} + \mathbf{i}\frac{\partial}{\partial\varepsilon} + \mathbf{i}\frac{\partial}{\partial\omega} + \mathbf{k}\frac{\partial}{\partial\chi}, & & \frac{\partial}{\partial\dot{\psi}^q} &=& \frac{\partial}{\partial\psi} - \mathbf{i}\frac{\partial}{\partial\varepsilon} - \mathbf{i}\frac{\partial}{\partial\omega} - \mathbf{k}\frac{\partial}{\partial\chi},\\
\frac{\partial}{\partial\psi^q_1} &=& \frac{\partial}{\partial\psi} + \mathbf{i}\frac{\partial}{\partial\varepsilon} + \mathbf{k}\frac{\partial}{\partial\chi}. & & \frac{\partial}{\partial\dot{\psi}^q_1} &=& \frac{\partial}{\partial\psi} - \mathbf{i}\frac{\partial}{\partial\varepsilon} - \mathbf{k}\frac{\partial}{\partial\chi}.
\end{array}$$

Here, ψ, φ, θ, ϕ, ς, χ, τ, ϵ, ε, ω are Euler angles of $Sp(1,1)$, $\theta^q = \theta+\phi-\mathbf{i}\tau$, $\varphi^q = \varphi-\mathbf{i}\epsilon+\mathbf{j}\varsigma$, $\psi^q = \psi - \mathbf{i}\varepsilon - \mathbf{i}\omega + \mathbf{k}\chi$ are quaternion Euler angles. The second Casimir operator W of $SO_0(1,4)$ is equal to zero on the representations of the class 1.

Matrix elements $t^\sigma_{mn}(\mathfrak{q}) = \mathfrak{M}^\sigma_{mn}(\varphi^q,\theta^q,\psi^q)$ of irreducible representations of the group $SO_0(1,4)$ are eigenfunctions of the operator (1):

$$\left[-F + \sigma(\sigma+3)\right]\mathfrak{M}^\sigma_{mn}(\mathfrak{q}) = 0, \tag{2}$$

where

$$\mathfrak{M}^\sigma_{mn}(\mathfrak{q}) = e^{-\mathbf{i}(m\varphi^q + n(\psi^q_1 - \mathbf{i}\omega))}\mathfrak{Z}^\sigma_{mn}(\cos\theta^q), \tag{3}$$

since $\psi^q = \psi^q_1 - \mathbf{i}\omega$. Here, $\mathfrak{M}^\sigma_{mn}(\mathfrak{q})$ are general matrix elements of the representations of $SO_0(1,4)$, and $\mathfrak{Z}^\sigma_{mn}(\cos\theta^q)$ are *hyperspherical functions*. Substituting the functions (3) into (2) and taking into account the operator (1), after substitution $z = \cos\theta^q$ we arrive at the following differential equation:

$$\left[(1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} - \frac{m^2+n^2-2mnz}{1-z^2} + \sigma(\sigma+3)\right]\mathfrak{Z}^\sigma_{mn}(z) = 0. \tag{4}$$

The latter equation has three singular points -1, $+1$, ∞. It is a Fuchsian equation. A particular solution of (4) can be expressed via the hypergeometric function

$$\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)=C_1\sin^{|m-n|}\frac{\theta^q}{2}\cos^{|m+n|}\frac{\theta^q}{2}\times$$
$$\times\,{}_2F_1\left(\left.\begin{matrix}\sigma+3+\frac{1}{2}(|m-n|+|m+n|),-\sigma+\frac{1}{2}(|m-n|+|m+n|)\\|m-n|+1\end{matrix}\right|\sin^2\frac{\theta^q}{2}\right). \qquad (5)$$

An explicit form of the functions $\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)$ can be derived via the multiple hypergeometric series. Namely, using an addition theorem for generalized spherical functions [3], we obtain

$$\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)=\sqrt{\frac{\Gamma(\sigma+m+1)\Gamma(\sigma-n+1)}{\Gamma(\sigma-m+1)\Gamma(\sigma+n+1)}}\cos^{2\sigma}\frac{\theta}{2}\cos^{2\sigma}\frac{\phi}{2}\cosh^{2\sigma}\frac{\tau}{2}\times$$
$$\sum^{\sigma}_{k=-\sigma}\sum^{\sigma}_{t=-\sigma}\mathbf{i}^{m-k}\tan^{m-t}\frac{\theta}{2}\tan^{t-k}\frac{\phi}{2}\tanh^{k-n}\frac{\tau}{2}\times$$
$${}_2F_1\left(\left.\begin{matrix}m-\sigma,-t-\sigma\\m-t+1\end{matrix}\right|-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\left.\begin{matrix}t-\sigma,-k-\sigma\\t-k+1\end{matrix}\right|-\tan^2\frac{\phi}{2}\right){}_2F_1\left(\left.\begin{matrix}k-\sigma,-n-\sigma\\k-n+1\end{matrix}\right|\tanh^2\frac{\tau}{2}\right) \qquad (6)$$

for $m\geq t,\ t\geq k,\ k\geq n$. In addition to (6) there exist seven functions $\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)$ for $m\geq t,\ k\geq t,\ k\geq n$; $t\geq m,\ k\geq t,\ n\geq k$; $t\geq m,\ t\geq k,\ n\geq k$; $t\geq m,\ k\geq t,\ k\geq n$; $t\geq m,\ t\geq k,\ k\geq n$; $m\geq t,\ t\geq k,\ n\geq k$; $m\geq t,\ k\geq t,\ n\geq k$.

Hyperspherical functions for other homogeneous spaces of $SO_0(1,4)$ are particular cases of the functions (6). For example, on the quaternion 2-sphere we have associated functions $\mathfrak{Z}^{m}_{\sigma}(\cos\theta^q)$. Further, the function (6) is reduced to the Jacobi function $\mathfrak{P}^{\sigma}_{mn}(\cosh\tau)$ on the hyperboloid $H^4\sim SO_0(1,4)/SO(4)$ and to a generalized spherical function $P^{\sigma}_{mn}(\cos\theta)$ on the real 3-sphere. Finally, on the surface of the real 2-sphere $S^2\sim SO(3)/SO(2)$ we have from (6) the usual spherical functions $Y^m_{\sigma}(\cos\theta)$.

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